

VANISHING CYCLES IN COMPLEX SYMPLECTIC GEOMETRY

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ABSTRACT. We study the vanishing cycles on the Milnor fibre for some non-isolated singularities which appear naturally in symplectic geometry. Under assumptions given in the text, we show that the vanishing cycles associated to a distinguish basis freely generate the corresponding homology groups of the Milnor fibre. We derive some consequences of this fact, in particular for the study of adjoint orbits in Lie algebras.

INTRODUCTION

The local Picard-Lefschetz theory studies the vanishing cycles associated to a holomorphic function germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}, 0)$ ([22, 24, 25]). The first case to be studied has been that of isolated singularities, then using the Whitney-Thom theory of stratified spaces and morphisms, Deligne, Grothendieck, Hironaka, Lê and others were able to develop a general theory which includes the case of non-isolated singularities ([9, 16, 17, 20]).

Using part of these works, we show that the non-isolated singularities which appear in symplectic geometry, as for instance in finite dimensional integrable systems, behave to a large extent like isolated hypersurface singularities. To this aim, we shall make an axiomatisation of these type of non-isolated singularities.

Our main result states that, under assumptions given in the text, the Lefschetz type vanishing cycles freely generate the corresponding homology group. This fact, which is already wrong for arbitrary isolated complete intersection singularities, shows that the non-isolated complete intersection singularities that we consider are, from the point of view of topology, the simplest one after the case of isolated hypersurface singularities.

Indeed, our approach is purely topological and elementary, close in spirit to the original works of Brieskorn, Lê, Milnor and Pham; it does not use the Deligne-Grothendieck formalism nor any sheaf theoretic methods. The philosophy is in some sense opposite: by elementary topological considerations,

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we hope deduce non-trivial facts on deformation theory and quantisation; this is however at the present stage only speculative. Although the applications outside the realm of differential topology are not firmly established, we can apply our theorem for investigating the topology of non-isolated singularities; we shall give two examples: involutive mappings in symplectic spaces (which includes integrable systems in symplectic spaces) and the Steinberg mapping associated to a semi-simple Lie algebra. The case of integrable systems is particularly relevant for perturbation theory since we now that the Lagrangian deformation theory is essentially topological ([13]). Most of the results that we obtained are part of the following program formulated by Arnold at his seminar ([3], 1992-2): “(...) Construct a monodromy and variation theory for non-isolated singularities (...). The obtained theory must apply to the mapping that assigns characteristic polynomials to matrices. It must generalize the Brieskorn-Grothendieck description of simple singularities for the A,D,E surfaces over non quasi-regular elements of Lie algebras.”

1. NON-ISOLATED COMPLETE INTERSECTION SINGULARITIES

We start by showing how non-isolated singularities typically appear in symplectic geometry; then we review classical facts on non-isolated complete intersection singularities.

1.1. Propagation of singularities in symplectic geometry. Let us consider the space $\mathbb{C}^{2n} = \{(q, p)\}$ together with the symplectic form $\sum_{i=1}^n dq_i \wedge dp_i$ and a holomorphic mapping $f = (f_1, \dots, f_k) : M \longrightarrow \mathbb{C}^k$ whose components are in involution ,i.e.,

$$\forall i, j \leq k, \quad \{f_i, f_j\} = \sum_{l=1}^n \partial_{q_l} f_i \partial_{p_l} f_j - \partial_{p_l} f_i \partial_{q_l} f_j = 0.$$

Here $M \subset \mathbb{C}^{2n}$, $S \subset \mathbb{C}^n$ denote open subsets.

The fibres of such an involutive mapping might be, for instance, characteristic varieties of partial differential equations depending on a semi-classical parameter.

The Hamiltonian vector fields associated to the components of f are tangent to its fibres, therefore such an involutive map may generically have fibres with a non-isolated singular locus.

As a simple example, consider the map $f = (f_1, f_2) : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ defined by the equations $f_1 = p_1 q_1$, $f_2 = p_2$ and denote by V_0 its fibre above the origin. The variety V_0 is analytically equivalent to the product of the plane curve $\Gamma = V_0 \cap \{q_2 = 0\}$ by an affine line. The singular locus of the variety V_0 is one dimensional and the Hamiltonian vector field $X = \partial_{q_2}$ associated to f_2 is tangent to it.

This example is stable in the sense that deforming the germ of f at the origin by a map whose components are in involution, leaves the analytical type of the singular fibration germ given by f unchanged ([11, 12]).

Remark that the smooth fibres of f have a vanishing topology which is drastically different from that of isolated complete intersection singularities. In such a case, by the classical results of Milnor and Hamm ([15, 24]), the Milnor fibre of the germ $f : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^2, 0)$ would have been homeomorphic to a bouquet of 2-dimensional spheres while it is homeomorphic to a circle. This is a general feature for singular Lagrangian varieties since the first Betti number of the fibre is equal to the dimension of the Lagrangian versal deformation space ([13]).

1.2. The Milnor fibration for non-isolated complete intersection singularities. Consider a morphism $f : X \rightarrow S$ between complex spaces over a smooth base. There is, a priori, no reason for f to define a topologically trivial fibration above its regular values, even if X is smooth; Thom's a_f -condition ensures such a result.

DEFINITION 1. A continuous map between Whitney stratified topological spaces $f : X \rightarrow S$, $X = \cup_\alpha X_\alpha$, $S = \cup_\alpha S_\alpha$ is called *stratified* if the restriction of f to each strata is a submersion.

For a given holomorphic map $f : X \rightarrow S$, the union of singular loci of its fibres form an analytic space called the *critical locus* of the map, its image under the map is called the *discriminant*. The discriminant is a priori only a variety and does not carry a canonical scheme structure (it is nevertheless very likely that for the particular cases we will study, it does carry such a structure analogous to that discovered in [29] for isolated singularities).

DEFINITION 2 ([30]). A stratified map $f : X \rightarrow S$, $X \subset \mathbb{R}^n$ satisfies the *condition a_f* if for any sequence of point (x_n) in a strata X_α converging to a point x lying in an adjacent strata $X_{\alpha'}$ such that the sequence $\text{Ker } df|_{X_\alpha}(x_n)$ converges, we have the inclusion $\text{Ker } df|_{X_{\alpha'}}(x) \subset \lim_{n \rightarrow \infty} \text{Ker } df|_{X_\alpha}(x_n)$. The map f satisfies the *strong a_f -condition* if in addition the complement of the critical set of f is a strata.

A function germ satisfies the a_f -condition if it admits a representative satisfying this condition. Denote by $B_\delta \subset \mathbb{C}^m$, the closed ball centred at the origin of radius δ .

DEFINITION 3. A *standard representative* $\bar{f} : X \rightarrow S$ of a holomorphic map germ $f : (Y, 0) \rightarrow (\mathbb{C}^k, 0)$, $Y \subset \mathbb{C}^m$, is a proper representative which satisfies the following conditions

- (1) there exists a surjective holomorphic map $g : Y \rightarrow S'$ and a compact neighbourhood $S \subset S'$ of the origin so that \bar{f} is the restriction of g to $X = g^{-1}(S) \cap B_\delta$, for some closed ball B_δ contained in Y ,
- (2) the map g satisfies the a_f -condition,
- (3) the fibres of g are transversal to the boundary of the ball B_δ ,
- (4) the spheres centred at the origin of radius $r \leq \delta$ are transversal to the fibre of g above the origin.

In the sequel, we shall abusively use the same letter for the germ and a standard representative of it. A holomorphic map germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ whose special fibre is the germ of a reduced complete intersection will be called a *complete intersection map germ*.

A small modification of Ehresmann's lemma gives the following result.

PROPOSITION 1. *Consider a standard representative $f : X \longrightarrow S$ of a complete intersection map germ satisfying the a_f -condition. The restriction of f above the open stratum defines a C^∞ locally trivial fibration.*

Example 1. Denote by B the closed unit ball in \mathbb{C}^4 . Put $X = \{(x, y, z, \lambda) \in B : (xy + \lambda)z = 1\}$ and consider the projection on the last factor

$$f : X \longrightarrow S, (x, y, z, \lambda) \mapsto \lambda.$$

We consider the Whitney stratification with only two strata: one is the interior of X , the other one is the boundary of X . The fibre of f above λ is diffeomorphic to the complement of the curve $\{xy = \lambda\}$ in the affine plane $\mathbb{C}^2 = \{(x, y)\}$; therefore for $\lambda = 0$ it is homotopic to a 2-torus. It is a simple exercise to see that the first Betti number of the other fibres equals 1 (one can also prove that they are actually homotopic to the complement of a circle in \mathbb{R}^3). Therefore the stratified map $f : X \longrightarrow S$ does not satisfy the strong a_f -condition.

The fibration associated by a standard representative of a holomorphic map germ $f : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ does not depend on the choice of the representative, it is called the *Milnor fibration* of f . The fibre of a standard representative above the origin is called the *special fibre*.

1.3. The local Lefschetz theorem.

THEOREM 1 ([20]). *Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$, $X \subset \mathbb{C}^m$, be the germ of a complex analytic map satisfying the a_f -condition. For any smooth Milnor fibre V , there exists an Zariski open subset Ω in the space of linear hyperplanes $H \subset \mathbb{C}^m$ such that the pair $(V, H \cap V)$ is N -connected with $N = \dim V - 1$.*

Remark 1. The theorem was originally stated for a smooth manifold X but this condition is used only to ensure that a standard representative of f satisfies the a_f -condition.

THEOREM 2 ([21, 17]). *A holomorphic function germ $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ defined on a reduced complex space X satisfies the a_f -condition.*

Using the two above-mentioned theorems, we get the following generalisation of a theorem due to Kato-Matsumoto [18].

COROLLARY 1. *Consider a complete intersection map germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ satisfying the a_f -condition and denote by s the dimension of the singular locus of the special fibre. The smooth fibres of a standard representative of $f : X \longrightarrow S$ are $(m - k - s - 1)$ -connected.*

Proof. The proof is completely similar to the one given by Chéniot and Lê for hypersurface singularities [8].

We use induction on the dimension s of the singular locus of the special fibre. For $s = 0$, the result is due to Milnor and Hamm ([15, 24]).

Assume that $s > 0$, the subset $\Omega \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1})$ for which the zero fibre X_π of the map $\pi \circ f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^{k-1}, 0)$ is a reduced complete intersection germ, is readily seen to be a Zariski open subset (in case $k = 1$, we take $X_\pi = \mathbb{C}^m$).

Take $\pi \in \Omega$ and chose f_j so that the ideal generated by the components of f is the same than that generated by the components of $\pi \circ f$ and f_j . Denote by $g : X_\pi \longrightarrow (\mathbb{C}, 0)$ the restriction of the function f_j to X_π . Theorem 2 implies that for a well chosen stratification, the map g satisfies the a_f -condition.

Theorem 1 implies in turn that for a generic hyperplane H and any smooth fibre V associated to a standard representative of g , the pair $(V, V \cap H)$ is $(m - k - 1)$ -connected. The induction assumption states that $V \cap H$ is $(m - k - s - 1)$ -connected, therefore V is also $(m - k - s - 1)$ -connected. This concludes the proof of the corollary. \square

2. THE MILNOR FIBRATION FOR T_n -TYPE MAPS

2.1. Transverse maps. Recall that

- (1) two continuous map germs $f, g : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ are called *topologically equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^m, 0) & \xrightarrow{f} & (\mathbb{C}^k, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}^m, 0) & \xrightarrow{g} & (\mathbb{C}^k, 0) \end{array}$$

where the vertical arrows are homeomorphisms.

- (2) a holomorphic map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$ belongs to the A_1 *singularity class* if its quadratic differential at the origin is definite.

The complex Morse lemma states that a map which belongs to the A_1 singularity class can always be reduced to a sum of squares after a holomorphic change of coordinates.

DEFINITION 4. A *suspension* of a map germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$ is a map germ of the type

$$\tilde{f} : (\mathbb{C}^n \times \mathbb{C}^l, 0) \longrightarrow (\mathbb{C}^k \times \mathbb{C}^s, 0), \quad (x, y) \mapsto (f(x), \pi(y))$$

where π is a linear surjective mapping.

The critical values for which the germ of f is topologically equivalent to a suspension of an A_1 singularity $g : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ in $(n + 1)$ variables will be called TA_1^n *critical values*.

DEFINITION 5. A complete intersection map germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ is called of T_n -type if it satisfies the strong a_f -condition and if for any standard representative $f : X \longrightarrow S$, there is an open dense subset of TA_1^n critical values in the discriminant for which the singular locus of the variety $f^{-1}(z)$ is connected.

Example 2. Any V or $(R - L)$ -versal unfolding of an isolated complete intersection singularity is a T_n -type mapping ([4]).

2.2. **Existence of Lefschetz vanishing cycles.** Let $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ be a T_n -type mapping. Choose a linear mapping $\pi : \mathbb{C}^k \longrightarrow \mathbb{C}^{k-1}$ whose fibre intersect the discriminant at TA_1^n critical values over some Zariski open subset $U \subset \mathbb{C}^{k-1}$. Consider a complex line $L \subset \mathbb{C}^k$ which is the preimage of a point in U .

Take a set of paths connecting a regular value $\varepsilon_0 \in (\partial S \cap L)$ of f to each of the critical values contained in L .

Around a point x of type TA_1^n , the map f is topologically equivalent to

$$\tilde{f} : (x) \mapsto \left(\sum_{j=1}^{n+1} x_j^2, x_{n+2}, \dots, x_k \right).$$

As f defines a locally trivial fibration above the complement of the discriminant, we may transport the n -dimensional *vanishing sphere*

$$S_{x,\varepsilon} = \{x \in \mathbb{R}^k, \tilde{f}(x) = (\varepsilon, 0, \dots, 0)\}$$

along the path connecting the critical value $f(x)$ to ε_0 , we get a sphere in V whose non-oriented isotopy class may a priori depend on the choice of the critical point.

LEMMA. Two (unoriented) spheres $\alpha, \beta \subset V$ which vanish at two critical points with the same critical value are isotopic.

Proof. Denote by $\Delta \subset M$ the singular locus of the singular fibre $V_z = f^{-1}(z)$, $z \in L \cap \Sigma$ at which the spheres α, β vanish.

The spheres α, β vanish at different critical points say $x, x' \in V_z$ having the same critical value z , i.e., $f(x) = f(x') = z$.

In a small neighbourhood U of x the proposition is obviously true. Cutting a path joining x to x' into small open subsets, so that in each open subset the spheres are isotopic, we get that the vanishing spheres at x and x' are isotopic after a convenient choice of their orientations. \square

DEFINITION 6. A T_n -map germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ is called *calibrated* there exists a differential form $\alpha \in H^n(V, \mathbb{C})$ on the Milnor fibre V of f such that for any sphere γ vanishing at a TA_1^n critical point, the period integrals $\int_\gamma \alpha$ is non-zero.

The calibration allows us to get rid of the ambiguity in the orientation of the vanishing sphere simply by choosing the orientation for which $\text{Re} \int_\gamma \alpha > 0$

if non-zero and $\text{Im} \int_{\gamma} \alpha > 0$ otherwise.

Therefore to each path connecting a critical value z to the base point ε_0 is associated a well-defined oriented isotopy classes of a sphere, the homology class of this sphere is called the *Lefschetz vanishing cycle* at the critical point. A set of path connecting the regular value to the critical values of f contained in L is called *weakly distinguished* and the corresponding vanishing cycles are called a *set of Lefschetz vanishing cycles* associated to these paths and to the linear mapping $\pi : \mathbb{C}^k \longrightarrow \mathbb{C}^{k-1}$. The number of cycles contained in such a collection is equal to the multiplicity of the discriminant at the origin.

2.3. The free basis theorem.

DEFINITION 7. A holomorphic map germ $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ is called *simplifiable*¹ if at a generic point of the singular locus on the special fibre it has rank $(k - 1)$.

In the definition *generic point* means that the property holds outside an analytic subspace of codimension one inside the singular locus of the special fibre.

THEOREM 3. Let $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ be a simplifiable calibrated T_n -map germ with Milnor fibre V . There exists a Zariski open subset $\Omega \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1})$ such that: for $\pi \in \Omega$, any set of Lefschetz vanishing cycles associated to a weakly distinguished basis of paths in \mathbb{C}^k freely generate the homology group $H_n(V)$.

Here and in the sequel, the homology is taken with \mathbb{Z} -coefficient.

Remark 2. Corollary 1 and the Hurewicz isomorphism imply that the groups $H_n(V)$ and $\pi_n(V)$ are isomorphic provided that $n > 1$.

Remark 3. In the particular case $k = 1$, we recover the fact that for isolated hypersurface singularities the Lefschetz vanishing cycles associated to a morsification of the function generate the group $H_{m-1}(V)$ ([24]).

COROLLARY 2. Under the assumptions of the theorem, the multiplicity at the origin of the discriminant m_{Σ} of the map-germ f is equal to the n -th Betti number of its Milnor fibre:

$$m_{\Sigma} = \text{rank } H_n(V).$$

2.4. **Proof of Theorem 3 (first part).** We prove that the vanishing cycles generate the homology group $H_n(V)$.

Denote by V_0 the special fibre of a standard representative $f : X \longrightarrow S$ above the origin, and by s the dimension of the singular locus of the variety V_0 , so that $n = m - k - s$. According to the results of Subsection 1.3, we may chose a codimension s linear space $P = \text{Ker}(u : \mathbb{C}^m \longrightarrow \mathbb{C}^s)$ such that

¹French word which means that can be simplified.

- (1) the pair $(V \cap P, V)$ is $(m - k - s)$ connected where s denotes the dimension of the singular locus of V_0 ,
- (2) the variety $V_0 \cap P$ is a reduced complete intersection with an isolated singular point at the origin.

The exact sequence of the pair $(V \cap P, V)$ implies that the inclusion $V \cap P \subset V$ induces a surjective mapping

$$\xi : H_n(V \cap P) \longrightarrow H_n(V).$$

Let us investigate the kernel of this map.

The vanishing cycles associated to (f, u) generate the middle-dimensional homology of the manifold $V \cap P$ ([15, 24]).

These vanishing cycles may be divided into two classes as follows. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{(f, u)} & S \times \mathbb{C}^s \supset \tilde{\Sigma} \\ & \searrow f & \downarrow p \\ & & S \supset \Sigma \end{array}$$

The discriminant of the map (f, u) has two components. One of the components $\tilde{\Sigma}$ is the preimage of the discriminant Σ of f under the projection on the first factor $p : S \times \mathbb{C}^s \longrightarrow S$. The other component correspond to regular values of f at which the affine space $\{u = \text{constant}\}$ is tangent to the Milnor fibre of f . The corresponding cycles are obviously null-homologous in V , i.e., they lie in the kernel of the map ξ .

At a critical value of f several distinct vanishing spheres of (f, u) may vanish. Lemma 2.2 implies that these spheres may be oriented so to define identical homology classes in V .

Thus, the image of the map ξ is generated by any set of Lefschetz vanishing cycles associated to f . It remains to prove that they do not satisfy any non-trivial relation.

2.5. Simplified mappings. The Zariski open subset Ω in the statement of Theorem 3 is defined in the following proposition.

PROPOSITION 2. *Let $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^k, 0)$ be a simplifiable complete intersection map germ. Assume that the special fibre of f has a singular locus of codimension $s > 0$. Then there exists a Zariski open subset $\Omega \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1})$ such that the special fibre of the map germ $\pi \circ f$ has a singular locus of codimension at least $s + 1$ for $\pi \in \Omega$.*

Proof. Consider a representative $f : X \longrightarrow S$. Take a Whitney stratification of the special fibre obtained by refining the stratification by the rank of f . Let X_{sing} be the open strata contained in the singular locus of the special fibre. At a point of X_{sing} the rank of f equals $k - 1$.

Denote by $Z = \{(\pi, x)\} \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1}) \times X_{\text{sing}}$ the analytic subvariety

for which the germ of $\pi \circ f$ has a critical point at x . The projections on the first and second factor give a diagram

$$\begin{array}{ccc} Z \subset \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1}) \times X_{\mathrm{sing}} & \xrightarrow{p_1} & X_{\mathrm{sing}} \\ \downarrow p_2 & & \\ \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{k-1}) & & \end{array}$$

As f is simplifiable, the variety $p_2(Z)$ is of positive codimension. For $\pi \notin p_2(Z)$, the map $\pi \circ f$ is regular along X_{sing} , therefore its critical space intersect its special fibre inside the boundary of \bar{X}_{sing} which has codimension $s + 1$. This proves the proposition. \square

2.6. Proof of Theorem 3 (second part). We choose a linear mapping $\pi : \mathbb{C}^k \rightarrow \mathbb{C}^{k-1}$ as in Proposition 2.

Denote by $A \subset M$ the Milnor fibre of the involutive pyramidal map $\pi \circ f$ so that $V \subset A$ (if $k = 1$ we take $A = M$).

Let us consider the exact sequence of the couple (A, V) :

$$\cdots \rightarrow H_j(V) \rightarrow H_j(A) \rightarrow H_j(A, V) \rightarrow \cdots$$

According to Corollary 1, we have the equalities

$$H_n(A) = 0, \quad H_{n+1}(A) = 0.$$

This gives an isomorphism

$$H_n(V) \simeq H_{n+1}(A, V).$$

Denote by $D_1, \dots, D_l \subset (T \cap L)$ closed disks centred at the critical values and containing no other critical value of f .

The union of the preimages under f of the disks D_1, \dots, D_l are the union of smooth connected manifolds, that we denote by $\hat{D}_1, \dots, \hat{D}_l \subset A$.

The excision isomorphism gives the decomposition

$$H_{n+1}(A, V) \simeq \bigoplus_{i=1}^l H_{n+1}(\hat{D}_i, V_i)$$

where V_i is the fibre of f at an arbitrary chosen point of the boundary of D_i .

The isomorphism $H_{n+1}(V) \simeq \bigoplus_{i=1}^l H_{n+1}(\hat{D}_i, V_i)$ sends each vanishing sphere γ_i to a ball C_i which bounds a vanishing sphere in V_i .

The ball C_i generates the group $H_{n+1}(\hat{D}_i, V_i)$ and it is non-zero since the map is calibrated. This concludes the proof of the theorem.

2.7. The Picard-Lefschetz formula for T_n type mappings. We keep the notations of the preceding subsection. The main observation is given by the following proposition.

PROPOSITION 3. *Under the assumptions of Theorem 3, there exists a finite group \mathcal{G} which acts on $H_n(V \cap P)$ preserving the intersection pairing so that $H_n(V \cap P)/\mathcal{G}$ is isomorphic to $H_n(V)$.*

Proof. Take two oriented vanishing spheres γ_1, γ_2 in $V \cap P$ which define the same homology class in V . The spheres γ_1, γ_2 vanish at critical points x_0, x_1 having the same critical value $z = f(x_0) = f(x_1)$.

Denote by V_z the fibre of f above $z \in S$ and take a C^∞ path inside the singular locus of V_z connecting x_0 to x_1 :

$$x : [0, 1] \longrightarrow V_z, \quad x(0) = x_0, \quad x(1) = x_1.$$

Fix points $p_1, p_2, \dots, p_{d-1} \in P$, $d = \dim P$ and denote by P_t the plane passing through $p_1, \dots, p_{d-1}, x(t), x(1-t)$. Transporting horizontally the homology class in $H_n(V \cap P_t)$ we get a transformation of the cycles which exchange γ_1 and γ_2 , fixes all vanishing cycles which do not vanish at z , and preserves the intersection pairing. This defines a finite group \mathcal{G} acting on $H_n(V \cap P)$. Theorem 3 implies that the quotient of $H_n(V \cap P)$ under the \mathcal{G} -action is isomorphic to $H_n(V)$. \square

The classical Picard-Lefschetz formula states that for a homology class $a \in H_n(V)$ in the Milnor fibre of an isolated hypersurface singularity, the monodromy around a Morse critical value is given by

$$h_* a = a + (a, \Delta) \Delta$$

where Δ is the cycle vanishing at the critical value ([22]). For non-isolated singularities such a formula does not make sense since the two cycles a and Δ do not intersect, if they are in general position.

PROPOSITION 4. *Under the assumptions of Theorem 3, there is a unique map D such that the diagram*

$$\begin{array}{ccc} H_n(V \cap P) & \xrightarrow{p} & H_n(V) \\ \downarrow P.D. & & \downarrow D \\ H^n(V \cap P) & \xleftarrow{i} & H^n(V) \end{array}$$

is commutative. Here p, i are the maps induced by the inclusion $V \cap P \subset V$ and $P.D.$ is the Poincaré duality. Moreover, the monodromy $h_ : H_n(V) \longrightarrow H_n(V)$ around a critical value is given by the Picard-Lefschetz type formula*

$$h_* = Id + D\Delta$$

where Δ is the cycle vanishing at the critical value.

Proof. According to Theorem 3, the map p is surjective and the map i is injective.

Take a lifting $s : H_n(V) \longrightarrow H_n(V \cap P)$ of the map p , i.e. $p \circ s = Id$. This lifting induces a dual mapping $s^* : H^n(V \cap P) \longrightarrow H^n(V)$.

According to Proposition 3, the map $D = s^* \circ (\cdot) \circ s$ is independent on s and satisfies the Picard-Lefschetz formula stated above. \square

PROPOSITION 5. *The variation operator $Var : H^n(V) \longrightarrow H_n(V)$ is well-defined and it is an isomorphism.*

Proof. We keep the notations used in the proof of the previous proposition. As $V \cap P$ is the Milnor fibre of an isolated complete intersection singularity, the variation operator $\widetilde{Var} : H^n(V \cap P) \longrightarrow H_n(V \cap P)$ exists (see e.g. [5]). We define the variation operator in $H^n(V)$ using the commutative diagram

$$\begin{array}{ccc} H^n(V) & \xhookrightarrow{i} & H^n(V \cap P) \\ \downarrow Var & & \downarrow \widetilde{Var} \\ H_n(V) & \xleftarrow{p} & H_n(V \cap P) \end{array}$$

According to Theorem 3, we may chose a basis of vanishing cycles which generate the group $H_n(V)$ freely. Take for $H^n(V)$ the dual basis to this basis. In these basis, the Picard-Lefschetz formula implies that the variation operator is triangular with diagonal elements equal to 1 or -1 (see e.g. [5], Chapitre I, section 2.5). This proves the proposition. \square

3. INVOLUTIVE MAPPINGS IN SYMPLECTIC SPACES

3.1. Pyramidal mappings. Let us now consider the case, where $\mathbb{C}^m = \mathbb{C}^{2n}$ is endowed with the holomorphic symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. The symplectic form induces a Poisson Bracket

$$\{f, g\}\omega^n = df \wedge dg \wedge \omega^{n-1}$$

which leads to the standard formula

$$\{f, g\} = \sum_{i=1}^n \partial_{q_i} f \partial_{p_i} g - \partial_{p_i} f \partial_{q_i} g.$$

The interior product with ω induces an isomorphism between the sheaves of holomorphic one-forms and that of holomorphic vector fields. Given a holomorphic function $H : U \longrightarrow \mathbb{C}$, defined on an open subset $U \subset \mathbb{C}^{2n}$, the vector field associated to the holomorphic one-form dH is called the *Hamiltonian vector field* associated to H .

DEFINITION 8. A holomorphic map $f : X \longrightarrow S$, $S \subset \mathbb{C}^k$ is called *involutive* if the Poisson brackets of its components vanish.

DEFINITION 9. A *pyramidal map* $f = (f_1, \dots, f_k) : U \longrightarrow \mathbb{C}^k$ where U is an open subset in \mathbb{C}^{2n} is a stratified involutive map such that for any critical point x lying on a strata X_α , the Hamiltonian vector fields v_1, \dots, v_k give local coordinates around x on $f^{-1}(x) \cap X_\alpha$, i.e.:

$$T_x(f^{-1}(x) \cap X_\alpha) = \text{Span}\{v_1(x), \dots, v_k(x)\}$$

at any point x of a strata S inside the singular locus of a fibre.

A similar notion holds for germs, in that case it is sufficient to check the property on the special fibre.

Example 3. For $k = 1$, a map is pyramidal provided it has isolated critical points.

Example 4. Let $f = (f_1, f_2) : (\mathbb{C}^4, 0) \longrightarrow (\mathbb{C}^2, 0)$ be an involutive complete intersection map germ. Assume that some linear combination of f_1 and f_2 has an isolated critical point at the origin, then f is pyramidal

3.2. Pyramidal involutive mappings satisfy the a_f -condition.

PROPOSITION 6. *Any involutive pyramidal map germ $f : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ on a symplectic space satisfies the strong a_f -condition. In particular, the restriction of the map f above the complement of the discriminant defines a C^∞ locally trivial fibration.*

Proof. We use the notations of Definition 1.

Let $f : X \longrightarrow S$ be a representative of the germ f satisfying the first two conditions of Definition 1 and denote by X_s the fibre above $s \in S$.

Chose a Whitney stratification of X obtained by a refinement of the stratification by the rank of f and whose intersection with the special fibre defines a Whitney stratification.

Let x be a singular point of the fibre X_0 , the pyramidity assumption implies that the tangent space to the strata of x is generated by the Hamiltonian vector fields associated to f_1, \dots, f_k . These Hamiltonian fields are defined in a neighbourhood of x and therefore by transversality the tangent space $T_x S$ at x is the limit of the tangent spaces to the adjacent fibres.

(Take a vector v_x tangent at x to the strata of x . The pyramidity assumption implies that there exist $a_1, \dots, a_n \in \mathbb{C}$ such that $v_x = \sum a_i v_i(x)$ where v_i is the Hamiltonian field associated to f_i . The vector field $\sum a_i v_i$ is globally defined on M and it is tangent to the fibres of f .) This proves the proposition. \square

3.3. Pyramidal mappings in symplectic spaces are calibrated. The proof of the following proposition is straightforward.

PROPOSITION 7. *Let $f : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ be the germ of an involutive map with a non-vanishing differential at the origin. There exist biholomorphic maps $\varphi : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^{2n}, 0)$, $\psi : (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0)$ such that φ is symplectic and*

$$\psi \circ f \circ \varphi = (p_1, \dots, p_j, g)$$

where j denotes the rank of $df(0)$.

The symplectic form ω being closed, we may chose a differential one-form α such that $d\alpha = \omega$. In local coordinates, we have $\alpha = \sum_{i=1}^n p_i dq_i$.

PROPOSITION 8. *Any pyramidal involutive T_n -mapping is calibrated with respect to the differential form $\alpha \wedge (d\alpha)^{n-k}$.*

Proof. According to the previous proposition at any smooth point of a fibre, we may chose local Darboux coordinates such that $f(q, p) = (p_1, \dots, p_k)$. These choice of coordinates shows that the restriction of the differential form $\alpha \wedge (d\alpha)^{n-k}$ to the fibres of f is closed and hence define a cohomology class.

Any vanishing sphere γ in the Milnor fibre is the boundary of a ball B . By Stokes formula, we get

$$\int_{\gamma} \alpha \wedge (d\alpha)^{n-k} = \int_B (d\alpha)^{n-k+1} \neq 0$$

This proves the proposition. \square

3.4. Arnold-Liouville manifolds. Consider the pyramidal involutive map

$$f : (\mathbb{C}^6, 0) \longrightarrow (\mathbb{C}^2, 0), (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (p_1 q_1 + p_2 q_2, p_2 q_2 + p_3 q_3).$$

The discriminant of f consists in the germ at the origin of the three lines $\{s_1 = 0\} \cup \{s_2 = 0\} \cup \{s_1 = s_2\}$. Therefore, the first three homotopy groups of the Milnor fibre V of f are $\pi_1(V) = \pi_2(V) = 0$ and $\pi_3(V) = H_3(V) = \mathbb{Z}^3$. So, the complex manifold V is a 2-connected affine 4-fold with non-vanishing three dimensional homology group.

This example can be generalised as follows. Let $g = (g_1, \dots, g_n) : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^n, 0)$ be the map germ defined by $g_i = p_i q_i$. Denote by $R : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ a surjective linear map such that $R \circ g$ is pyramidal. The Milnor fibre of the pyramidal involutive map $f = R \circ g : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ is a $2(n-k)$ -connected $2n-k$ -dimensional complex affine manifold whose Betti number $b_{2(n-k)+1}$ equals the binomial coefficient $\binom{n}{k-1}$. In case $k = n$, the fibres are homotopic to Arnold-Liouville tori, therefore we call such manifolds arising from integrable system *Arnold-Liouville manifolds*.

3.5. The Henon-Heiles integrable system.

PROPOSITION 9. *The Milnor fibre V of the Henon-Heiles integrable system $H = (H_1, H_2) : (\mathbb{C}^4, 0) \longrightarrow (\mathbb{C}^2, 0)$ with*

$$H_1 = p_1^2 + p_2^2 - 4q_2^3 - 2q_1^2 q_2, \quad H_2 = q_1^4 - 4q_1^2 q_2^2 + 4p_1(q_1 p_2 - q_2 p_1)$$

has its first Betti number equal to 4.

Proof. The discriminant of the Henon-Heiles integrable system

$$H_1 = p_1^2 + p_2^2 - 4q_2^3 - 2q_1^2 q_2, \quad H_2 = q_1^4 - 4q_1^2 q_2^2 + 4p_1(q_1 p_2 - q_2 p_1)$$

has two components. One of the component is the curve $s_2^3 = s_1^4$ the other component is the straight line $\{s_2 = 0\}$ ([14]).

The fibres of the map $f = (H_1, H_2) : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ are open parts of Prym varieties associated to the two fold covering of an elliptic curve by a hyperelliptic genus 3 curve ([14, 23]). Therefore the singular locus of a generic singular fibre is connected, this can also be seen directly: a direct computation shows that the critical locus of f has two components, each component intersecting the special fibre along a reduced connected curve. Applying Theorem 3, we get that the first Betti number of the Milnor fibre is equal to 4. This proves the proposition. \square

4. THE VANISHING TOPOLOGY OF COADJOINT ORBITS

4.1. The Steinberg mapping of a semi-simple Lie algebra. A *Poisson structure* in \mathbb{C}^n is an antisymmetric biderivation

$$\{\cdot, \cdot\} : \mathcal{O}_{\mathbb{C}^n} \times \mathcal{O}_{\mathbb{C}^n} \longrightarrow \mathcal{O}_{\mathbb{C}^n}$$

satisfying the Jacobi identity.

A Poisson structure on a finite dimensional vector space V is given by specifying the values of the Poisson bracket on the space of linear forms V^* . By identifying a Lie algebra with its bidual we see that the Lie Bracket defines a Poisson bracket on the dual of the Lie algebra

$$\{a, b\} := [a, b]$$

called the *Kirillov-Kostant-Souriau Poisson structure*. This Poisson structure defines a symplectic structure on coadjoint orbits, that is, identifying \mathfrak{g}^* with the image of a faithful representation, the orbits under conjugacy. The *Steinberg mapping* $S : \mathfrak{g} \longrightarrow \mathfrak{h}/W$ is the map sending an element $x = x_s + x_n$ to the class of its semi-simple part (in the Jordan decomposition) modulo the action of the Weyl group. By Chevalley's theorem there is a scheme isomorphism of the Cartan subalgebra modulo the Weyl group and the adjoint quotient

$$\mathfrak{h}/W \approx \mathfrak{g}/G.$$

By Cartan's theorem, as \mathfrak{g} is simple, the Killing form \mathfrak{g} is non-degenerate and therefore coadjoint orbits and adjoint orbits can be identified.

PROPOSITION 10. *The germ of the Steinberg mapping at any point is a simplifiable calibrated T_2 -mapping.*

4.2. Some remarks concerning the vanishing topology of coadjoint orbits. Proposition 10 enables us to apply Theorem 3 for the Steinberg mapping. Remark that in particular the Steinberg mapping defines a C^∞ locally trivial fibration above the complement of its discriminant. We combine this result with the following theorem due to Brieskorn.

THEOREM 4 ([6]). *Denote by $(m_{ij})_{i,j=1,\dots,r}$ the Coxeter matrix associated to a semi-simple Lie algebra \mathfrak{g} . Then, there exists homotopy classes b_1, \dots, b_r associated to a basis of Lefschetz vanishing cycles such that the group $\pi_1(\mathfrak{h}/W \setminus \Sigma)$ admits a presentation with generators b_1, \dots, b_r and relations*

$$\underbrace{b_i b_j b_i \dots}_{m_{ij} \text{ times}} = \underbrace{b_j b_i b_j \dots}_{m_{ij} \text{ times}} \text{ whenever } m_{ij} \geq 3$$

By adding the relations $b_i^2 = 1$ to the group $\pi_1(\mathfrak{h}/W \setminus \Sigma)$, we get the *Coxeter group* G associated to the Lie algebra. The group G admits a faithful representation as a finite reflexion group in \mathbb{R}^n . The induced representation on $\pi_1(\mathfrak{h}/W \setminus \Sigma)$ is called the *Coxeter representation*.

According to the standard Picard Lefschetz formula and to its version for non-isolated singularities (see Subsection 2.7), the monodromy of Lefschetz

vanishing cycles is a reflection in the hyperplane orthogonal to the vanishing cycle. By Theorem 3, all these hyperplane are distinct and Brieskorn's theorem gives the angle between this hyperplanes, therefore we get the following result.

THEOREM 5 (Compare [26]). *The monodromy representation*

$$\pi_1(\mathfrak{h}/W \setminus \Sigma) \longrightarrow GL(r, H_2(V) \otimes \mathbb{R}),$$

associated to the germ of the Steinberg mapping of a simple Lie algebra at a critical point x with $S(x) = 0$, coincides with the Coxeter representation.

Applying the results of Subsection 2.7 and comparing this Theorem with the results of Brieskorn, Esnault and Slodowy ([7, 10, 27]), we get that, at a subregular point, for the simple Lie algebras of type A, D, E there is an isomorphism $H_2(V \cap P) \approx H_2(V)$ and that the group \mathcal{G} is trivial, while for the ones of type B, C, F, G the group \mathcal{G} are respectively the group $\mathbb{Z}/2\mathbb{Z}$ for B, C, F and the permutation group \mathcal{S}_3 for the group G_2 . These groups attached to the B, C, F, G Lie algebras were discovered independently by Arnold and Slodowy ([2] and [27], Chapter 6.2).

4.3. The a_f -condition and Casimir functions. Recall that a holomorphic function is called a *Casimir function* if it Poisson commutes with any other holomorphic function.

PROPOSITION 11. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$ be a complete intersection map germ such that the components of f are Casimir functions, then f satisfies the strong a_f -condition.*

Proof. The proof is similar to that of pyramidal involutive mappings as any vector v_x tangent to a strata of the special fibre at a point x extends to a Hamiltonian vector field in a neighbourhood of x . Indeed, choose a function $h : U \longrightarrow \mathbb{C}$ such that v_x is the Hamiltonian vector field v of h evaluated at x in some neighbourhood U of the point x . As the components of f Poisson commutes with h , the vector field v is tangent to the fibres of f . \square

Applied to the Steinberg mapping the proposition shows that it defines a locally trivial fibration in any small open ball containing the origin.

4.4. Proof of Proposition 10. An element $x \in \mathfrak{g}$ is called *regular* if its centraliser has dimension $r + 2$ where r is the rank of the Lie algebra, i.e, the dimension of a Cartan subalgebra. A theorem due to Kostant and Steinberg states that the non-regular elements are the critical values of the Steinberg mapping ([19, 28]).

An element $x \in \mathfrak{g}$ is called *subregular* if its centraliser has dimension $r + 2$. Any singular fibre of the Steinberg mapping contains a subset of subregular elements which is dense in the singular locus of the fibre ([28]). The rank of the Steinberg mapping at such point equals $r - 1$ ([7, 27]).

LEMMA. *Let $x \in \mathfrak{g}$ be a critical point of the Steinberg mapping. If $S(x)$ is a smooth point of the discriminant then x is semisimple and subregular.*

Proof. We consider the Jordan decomposition $x = x_s + x_n$ it into semisimple and nilpotent part and use the criterion ([28]):

x is subregular in $\mathfrak{g} \iff x_n$ is subregular in $C(x_s)$.

Here $C(x_s)$ denotes the centraliser of x_s .

Chose a Cartan subalgebra \mathfrak{h} containing x_s and let Φ be the associated root system. The root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

shows that $C(x_s) = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$ where α is the root which annihilates x_s ; thus x_s is subregular. As x_n is subregular in $C(x_s)$ and $C(x_s)$ has dimension $r + 2$ we get that $C(x_n) = C(x_s)$; in particular x_n commutes to all element in the Cartan subalgebra \mathfrak{h} , thus $x_n \in \mathfrak{h}$ and consequently $x_n = 0$. \square

The lemma implies in particular that the singular locus of $S^{-1}(S(x))$ is connected for x in the smooth locus of the discriminant.

LEMMA. (Compare [1]) *If x is a subregular semisimple element of the rank r Lie algebra \mathfrak{g} , then there exists a commutative diagram of holomorphic map germs*

$$\begin{array}{ccc} (\mathfrak{g}, x) & \xrightarrow{S} & (\mathfrak{h}/W, S(x)) \\ \downarrow \varphi & & \downarrow \psi \\ (\mathfrak{sl}(2, \mathbb{C}), 0) \times (\mathbb{C}^{r-1}, 0) & \xrightarrow{(det, Id)} & (\mathbb{C}, 0) \times (\mathbb{C}^{r-1}, 0) \end{array}$$

Proof. Like in the previous lemma, we consider the Chevalley-Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi'} \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}, \quad \Phi' = \Phi \setminus \{\beta, -\beta\}.$$

so that the roots $\pm\beta$ annihilate $x \in \mathfrak{h}$.

The tangent space to the adjoint orbit of x is

$$[x, \mathfrak{g}] = \bigoplus_{\alpha \in \Phi'} \mathfrak{g}_\alpha$$

therefore the vector space $\mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$ is transversal to it. Consequently, the germ at x of the Steinberg mapping factorises

$$\begin{array}{ccc} (\mathfrak{g}, x) & \xrightarrow{S} & (\mathfrak{h}/W, S(x)) \\ \downarrow & \nearrow f & \\ (\mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}, x) & & \end{array}$$

Consider the hyperplane $\mathfrak{h}_1 = \text{Ker} \beta$ and consider a supplementary one dimension vector space to it $\mathfrak{h}_2 \subset \mathfrak{h}$, the rank one Lie algebra $\mathfrak{h}_2 \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, therefore we get an isomorphism of Lie algebras

$$\mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} \approx \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{h}_1.$$

Via this identification, the Lie subgroup $H \subset G$ which fix the transversal act on $\mathfrak{sl}(2, \mathbb{C})$ by conjugation. Moreover, by Chevalley's theorem the action of G on \mathfrak{h}_1 is induced by that of the Weyl group and by Kostant-Steinberg characterisation of regular elements, the Steinberg mapping restricted to \mathfrak{h}_1 has maximal rank at x . This concludes the proof of the lemma. \square

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